JOURNAL OF MATERIALS SCIENCE **40** (2005) 5883–5892

Overall ultimate yield surface of periodic tetrakaidecahedral lattice with non-symmetric material distribution

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A rigorous method for the homogenization of general elastoplastic periodic lattices is applied to 3D cellular solids. Tetrakaidecahedral unit cell problems are solved to determine the overall yield surface of foams. Non-symmetric material distribution is introduced and new results concerning the influence of this type of defect are obtained. They show that non-uniform material distribution increases the overall strength, except in particular loading directions and that non-symmetry has no significant influence on the yield surface.

1. Introduction

Since the pioneering work on the mechanics of cellular solids by Gent and Thomas [1] as well as by Patel and Finnie [2] appeared, much work has been performed to an appropriate modelling of the effective elastic-perfectly plastic behavior of solid foams. Comprehensive treatises on elastic-perfectly plastic of solid foams are found in textbooks such as the well-known work by Gibson and Ashby [3]. Using simple beam theory, Klintworth and Stronge [4] proposed failure envelopes for regular honeycombs with respect to various elastic and plastic cell crushing modes. Gibson *et al*. [5] studied the biaxial yield surface of 2-dimensional honeycombs and the triaxial yield surface of 3-dimensional open-celled foams. In most of these studies, an upper bound on the plastic collapse stress is given by equating the work done by the applied stress to the plastic work done at the hinges corresponding to the considered collapse mode. More recently, Kim and Al-Hassani [6] developped an anisotropic hexagonal model to show the effects of strut morphology on plastic yield surface. Chen *et al*. [7] studied the influence of six different types of geometrical imperfection on the ultimate strength of 2D cellular solids. In these both cases, studied defects are symmetric and all the nodes of the lattice have the same weight.

The homogenization of periodic cellular solids with general cellular geometry and topology has been already presented in [8] and applied to 2D cellular solids with non-symmetric material distribution. The overall yield surface of cellular materials seen as periodic lattices of elastic-perfectly plastic beams that are rigidely connected in vertices is determined by solving a discrete yield design problem attached to the unit cell. This homogenization approach extends to limit analysis the method which has been previously presented in Sab [9], Pradel and Sab [10], Pradel [11] and Laroussi *et al*. [12], for elastic lattices.

The purpose of this work is to extend to 3D lattices the analysis for 2D cellular solids presented in [8]. The outline is as follows. Presented in Section 2 is a summary of the general homogenization method proposed in [8]. In Section 3, the symmetry of lattices is studied and used to simplify the unit cell problem. Section 4 is dedicated to the identification of the overall strength of a 3D Euler-Bernouilli beam with non-uniform section. Then, the discrete unit cell problem is set and solved in Section 5, for the determination of the overall strength properties of a tetrakaidecahedral lattice. The influence of non-symmetric defects is studied. The presentation is then concluded in Section 6 with a short summary of the results.

2. The static homogenization method for yield design of periodic discrete media [8]

The purpose of this section is to recall the static method for the determination of the macroscopic strength domain of general periodic lattices. Cellular materials seen as periodic lattices of beams that are rigidely connected in vertices will be considered in this paper. For these materials, particles are vertices of the lattice and interacting particles are couples of vertices which are connected by a beam element. The static method, which has been presented in [8] and applied to honeycomb

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DOI: 10.1007/s10853-005-5023-8

materials, is based on the resolution of a unit cell problem involving a finite number of periodic interaction forces and moments between the particles of the lattice. One can find in [13] a similar approach in the context of homogenization for granular materials. It is a generalization of the well-known homogenization method for the determination of the macroscopic strength domain of a continuum heterogeneous material which has been initially developped by Suquet [14], de Buhan [15] and Bouchitté [16] for periodic media, and by Sab [17] for random media.

Particles $P^{\alpha,i}, \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}^3, i = 1, \ldots, n$, of the so-called *n-type* lattice are generated by periodicity with n particles as follows:

$$\forall i = 1, \ldots, n, \ \forall_\alpha \in \mathbb{Z}^3,$$
$$\underline{X}^{\alpha,i} = \underline{X}^{0,i} + \alpha_1 \underline{a}_1 + \alpha_2 \underline{a}_2 + \alpha_3 \underline{a}_3$$

where $\underline{X}^{\alpha,i}$ is the coordinates vector of $P^{\alpha,i}$ in the reference configuration, and $\underline{a}_1, \underline{a}_2$ and $\underline{a}_3$ are three vectors forming a base in the 3-dimensional Euclidean space. $|Y| = |\det(a_1, a_2, a_3)|$ denotes the volume of the parallelepipedic cell constructed from this base, and $\mathcal{P}$ denotes the set of all particles of the lattice. It is assumed that, for every particle $P \in \mathcal{P}$, there exists a unique couple of $\alpha \in \mathbb{Z}^3$ and $i = 1, \ldots, n$ such that $P = P^{\alpha,i}$.

Let $c = \{P^{\beta,j}, P^{\alpha,i}\}$ be an interacting couple of particles: $\underline{f}_c^{\alpha,i}$(resp. $\underline{f}_c^{\beta,j}$) is the force exerted by particle $P^{\beta,j}$ (resp. $P^{\alpha,i}$) on particle $P^{\alpha,i}$ (resp. $P^{\beta,j}$), and $\underline{m}_c^{\alpha,i}$ (resp. $\underline{m}_c^{\beta,j}$) is the moment at point $\underline{X}^{\alpha,i}$ (resp. $\underline{X}^{\beta,j}$) exerted by particle $P^{\beta,j}$ (resp. $P^{\alpha,i}$) on particle $P^{\alpha,i}$ (resp. $P^{\beta,j}$). Fig. 1. The interaction forces and moments of couple c, noted

$$I_c = \left\{\underline{f}_c^{\beta,j}, \underline{m}_c^{\beta,j}, \underline{f}_c^{\alpha,i}, \underline{m}_c^{\alpha,i}\right\}$$

are self-balanced:

$$\underline{f}_c^{\alpha,i} + \underline{f}_c^{\beta,j} = 0$$
$$\underline{m}_c^{\alpha,i} + \underline{m}_c^{\beta,j} - \underline{f}_c^{\alpha,i} \wedge \underline{l}_c^{\alpha,i} = 0 \tag{1}$$

or, equivalently,

$$\underline{f}_c^{\alpha,i} + \underline{f}_c^{\beta,j} = 0$$
$$\underline{\widehat{m}}_c^{\alpha,i} + \underline{\widehat{m}}_c^{\beta,j} = 0$$

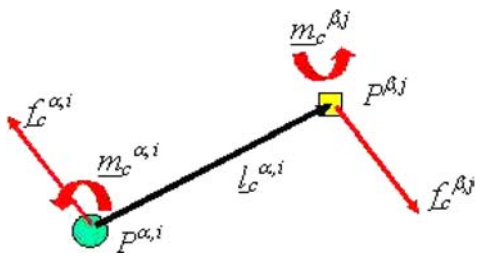

Figure 1 Interaction forces and moments.

where

$$\underline{l}_c^{\alpha,i} = \underline{X}^{\beta,j} - \underline{X}^{\alpha,i}$$
$$\underline{l}_c^{\beta,j} = \underline{X}^{\alpha,i} - \underline{X}^{\beta,j} = -\underline{l}_c^{\alpha,i}$$

are the branch vectors, and $\underline{\widehat{m}}_c^{\alpha,i}$ (resp. $\underline{\widehat{m}}_c^{\beta,j}$) is the moment at the mid-point of $P^{\alpha,i}$ and $P^{\beta,j}$ which is exerted by particle $P^{\beta,j}$ (resp. $P^{\alpha,i}$) on particle $P^{\alpha,j}$ (resp. $P^{\beta,j}$).

$$\underline{\widehat{m}}_c^{\alpha,i} = \underline{m}_c^{\alpha,i} - \frac{1}{2}\underline{f}_c^{\alpha,i} \wedge \underline{l}_c^{\alpha,i}$$
$$\underline{\widehat{m}}_c^{\beta,j} = \underline{m}_c^{\beta,j} - \frac{1}{2}\underline{f}_c^{\beta,j} \wedge \underline{l}_c^{\beta,j}$$

Depending on the orientation of couple c, which must be fixed once for all, the branch vector $\underline{l}_c$ and the generalized stress $(\underline{f}_c, \underline{\widehat{m}}_c)$ associated to c can be equivalently defined as:

$$\left\{\underline{l}_c^{\alpha,i} = \underline{l}_c, \ \begin{pmatrix} \underline{f}_c^{\alpha,i} = \underline{f}_c \\ \underline{\widehat{m}}_c^{\alpha,i} = \underline{\widehat{m}}_c \end{pmatrix}\right\}$$
$$\text{or} \quad \left\{\underline{l}_c^{\beta,j} = \underline{l}_c, \ \begin{pmatrix} \underline{f}_c^{\beta,j} = \underline{f}_c \\ \underline{\widehat{m}}_c^{\beta,j} = \underline{\widehat{m}}_c \end{pmatrix}\right\}$$

It will be assumed in the sequel that $\mathcal{C}$, the set of interacting couples of particles, is generated by periodicity with r couples noted $\{c_1, c_2, \ldots, c_r\}$. More precisely, for $\gamma \in \mathbb{Z}^3$ and $c = \{P^{\beta,j}, P^{\alpha,i}\}$, let $c^\gamma = \{P^{\beta+\gamma,j}, P^{\alpha+\gamma,i}\}$ denote the interacting couple of particles obtained by γ-translation of c. Then,

$$\mathcal{C} = \left\{c_k^\gamma, k = 1, \ldots, r, \gamma \in \mathbb{Z}^3\right\}$$

It is assumed that for every c in $\mathcal{C}$, there exists a unique couple of $\gamma \in \mathbb{Z}^3$ and $k = 1, \ldots, r$ such that $c = c_k^\gamma$. In the absence of external forces and moments, the balance equation at particle $P^{\alpha,i}$ writes:

$$\sum_{c \in \mathcal{C}} \underline{f}_c^{\alpha,i} = 0 \quad \text{and} \quad \sum_{c \in \mathcal{C}} \underline{m}_c^{\alpha,i} = 0 \tag{2}$$

In the above summations, $\underline{f}_c^{\alpha,i}$ and $\underline{m}_c^{\alpha,i}$ are zero if particle $P^{\alpha,i}$ is not one of the two particles of c.

The orientations of all interacting couples being fixed once for all, and at every interacting couple of particles c in $\mathcal{C}$, the closed nonempty convex domain of $\mathbb{R}^3 \times \mathbb{R}^3$, G^c, characterizing the strength capacities of this couple, is introduced:

$$(\underline{f}_c, \underline{\widehat{m}}_c) \in G^c$$

Let Γ denotes the application which associates G^c to every oriented couple c in $\mathcal{C}$. Assuming the periodicity of this application:

$$G^{c_k^\gamma} = G_k^c \ \text{for all } k \ \text{ and all } \gamma$$

it is possible to replace in the structural analysis the periodic lattice by a homogeneous continuum "effective" material with yet unknown strength domain. The task of the homogenization analysis is the determination of the macroscopic strength domain of the effective material.

For periodic interaction forces and moments in the infinite lattice such that

$$I_{c_k^\gamma} = I_{c_k} \quad \text{for all } k \text{ and all } \gamma, \tag{3}$$

the balance Equations (2) at particles $P^{\alpha,i}$ and $P^{0,i}$ are the same. Moreover, k and i being fixed, there exists at most one $\gamma \in \mathbb{Z}^3$ such that $P^{0,i}$ is one of the two particles of c_k^γ, and in this case $(\underline{f}^{0,i}_{c_k^\gamma}, \underline{m}^{0,i}_{c_k^\gamma})$ is noted $(\underline{f}^i_k, \underline{m}^i_k)$. Otherwise, $(\underline{f}^i_k, \underline{m}^i_k)$ is zero. Hence, (2) writes, for $i = 1, \ldots, n$:

$$\sum_{k=1,\ldots,r} \underline{f}^i_k = 0 \text{ and } \sum_{k=1,\ldots,r} \underline{m}^i_k = 0 \tag{4}$$

Let $\underline{\underline{\Sigma}}$ be the overall symmetric second order stress tensor applied to the infinite lattice. The set of statically admissible periodic interactions forces and moments associated to $\underline{\underline{\Sigma}}$ is:

$$\mathcal{SA}(\underline{\underline{\Sigma}}) = \Bigg\{ (I_c)_{c\in\mathcal{C}} / (3), (4) \text{ and } \underline{\underline{\Sigma}} = \frac{1}{|Y|} \sum_{k=1,\ldots,r} \underline{f}_{ck} \otimes^s \underline{l}_{-c_k} \Bigg\} \tag{5}$$

where $(\underline{a} \otimes^s \underline{b})_{ij} = \frac{1}{2}(a_i b_j + a_j b_i)$ is the symmetric part of the dyadic product of $\underline{a}$ and $\underline{b}$.

The macroscopic strength domain, denoted by $\mathcal{G}^{\text{hom}}$, is the convex domain of macroscopic stress states $\underline{\underline{\Sigma}}$ such that there exists a periodic distribution of interaction forces and moments $(I_c)_{c\in\mathcal{C}}$ in $\mathcal{SA}(\underline{\underline{\Sigma}})$, with $(\underline{f}_{c_k}, \underline{\widehat{m}}_{c_k})$ in G^{c_k} for all k:

$$\mathcal{G}^{\text{hom}} = \Bigg\{ \underline{\underline{\Sigma}} / \ \exists (I_c)_{c\in\mathcal{C}} \in \mathcal{SA}(\underline{\underline{\Sigma}}), (f_{c_k}, \underline{\widehat{m}}_{c_k}) \in G^{c_k} \text{ for all } k \Bigg\} \tag{6}$$

3. Material symmetry

As it has been seen above, the lattice is completely characterized by $(\mathcal{P}, \mathcal{C}, \Gamma)$ and the determination of $\mathcal{G}^{\text{hom}}$ requires the resolution of the unit cell problem (6). When the periodic lattice possesses a material symmetry property, the number of unknowns to be determined can be reduced as follows. Let $\underline{\underline{O}}$ be an orthogonal second order tensor: $\underline{\underline{O}}^{-1} = {}^t\underline{\underline{O}}$. For det $(\underline{\underline{O}})$ equal to $+1$, $\underline{\underline{O}}$ is a rotation on $\mathbb{R}^3$. The image by $\underline{\underline{O}}$ of particle $P \in \mathcal{P}$ of coordinates vector $\underline{X}$ is particle $P^\sharp$ of coordinates vector $\underline{\underline{O}}.\underline{X}$. $\mathcal{P}^\sharp = \{P^\sharp, P \in \mathcal{P}\}$ denotes the image of $\mathcal{P}$ by $\underline{\underline{O}}$. Similarly, the images $c^\sharp$ and $C^\sharp$ of $c \in \mathcal{C}$ and $\mathcal{C}$ are respectively defined. By definition of $G^{c^\sharp}$, $(\underline{f}_{c^\sharp}, \underline{\widehat{m}}_{c^\sharp}) \in G^{c^\sharp} \Leftrightarrow (\underline{f}_c, \underline{\widehat{m}}_c) \in G^c$, where $\underline{f}_{c^\sharp} = \underline{\underline{O}}.\underline{f}_c$ and $\underline{\widehat{m}}_{c^\sharp} = \det(\underline{\underline{O}})\underline{\underline{O}}.\underline{\widehat{m}}_c$ are the images of the force vector and the moment pseudo-vector associated to c, respectively. It can be easily seen from (6) that the macroscopic strength domain $(\mathcal{G}^{\text{hom}})^\sharp$ associated to $(\mathcal{P}^\sharp, \mathcal{C}^\sharp, \Gamma^\sharp)$ is such that $\underline{\underline{\Sigma}}^\sharp \in (\mathcal{G}^{\text{hom}})^\sharp \Leftrightarrow \underline{\underline{\Sigma}} \in \mathcal{G}^{\text{hom}}$ where $\underline{\underline{\Sigma}}^\sharp = \underline{\underline{O}}.\underline{\underline{\Sigma}}.{}^t\underline{\underline{O}}$ is the image of $\underline{\underline{\Sigma}}$. Moreover, $(I_c)_{c\in\mathcal{C}}$ is associated to $\underline{\underline{\Sigma}} \in \mathcal{G}^{\text{hom}}$ by (6) if, and only if, its image $(I^\sharp_{c^\sharp})_{c^\sharp\in\mathcal{C}^\sharp}$ is associated to $\underline{\underline{\Sigma}}^\sharp \in (\mathcal{G}^{\text{hom}})^\sharp$. By definition, $\underline{\underline{O}}$ is a material symmetry of the periodic lattice if $(\mathcal{P}^\sharp, \mathcal{C}^\sharp, \Gamma^\sharp) = (\mathcal{P}, \mathcal{C}, \Gamma)$. It results that $(\mathcal{G}^{\text{hom}})^\sharp = \mathcal{G}^{\text{hom}}$, which means that $\underline{\underline{\Sigma}}^\sharp$ is in $\mathcal{G}^{\text{hom}}$ if, and only if, its image $\underline{\underline{\Sigma}}^\sharp$ is in $\mathcal{G}^{\text{hom}}$. Now, if the material symmetry $\underline{\underline{O}}$ is such that $\underline{\underline{O}}^{(m)} = \text{Identity}$, then for invariant $\underline{\underline{\Sigma}} = \underline{\underline{\Sigma}}^\sharp$, one can restrict the analysis in (6) to invariant periodic forces and moments. Indeed let $(I_c)_{c\in\mathcal{C}}$ be associated to $\underline{\underline{\Sigma}} = \underline{\underline{\Sigma}}^\sharp$. Define $(I^\flat_c)_{c\in\mathcal{C}}$ as the arithmetic average of $(I_c)_{c\in\mathcal{C}}$ and its images by $\underline{\underline{O}}, \underline{\underline{O}}^{(2)}, \ldots, \underline{\underline{O}}^{(m-1)}$. Obviously, $(I^\flat_c)_{c\in\mathcal{C}}$ is in $\mathcal{SA}(\underline{\underline{\Sigma}})$ and it is invariant with respect to $\underline{\underline{O}}$. In addition, $(\underline{f}_c, \underline{\widehat{m}}_c)$ and all its images being in G^c, $(f^\flat_c, \underline{\widehat{m}}^\flat_c)$ is also in G^c by convexity. Therefore, $(I^\flat_c)_{c\in\mathcal{C}}$ can be associated to $\underline{\underline{\Sigma}} = \underline{\underline{\Sigma}}^\sharp$ in (6).

4. Ultimate yield strength of non-uniform 3D beam

Contained in this section is a discussion concerning the yield strength which is associated to a current beam of the lattice: G^c is computed in terms of the geometry and the material properties of the beam.

4.1. General 3D Euler-Bernouilli beam model

Consider a beam of length l and and non-uniform section $S(s)$ with $-\frac{l}{2} \leq s \leq \frac{l}{2}$, as shown in Fig. 2. Under the action of the forces and moments at the ends of the beam (i.e., at points $A : s = -\frac{l}{2}$, and $B : s = \frac{l}{2}$ in Fig. 2), it is assumed that the axial stress $\sigma_{ss}(s, y, z)$ is the only non negligible stress component. The axial force $N(s)$ and the bending moments $M_y(s)$ and $M_z(s)$ are:

$$N(s) = \int_{S(s)} \sigma_{ss}(s, y, z)\, dy dz \tag{7}$$

$$M_y(s) = \int_{S(s)} z\sigma_{ss}(s, y, z)\, dy dz \tag{8}$$

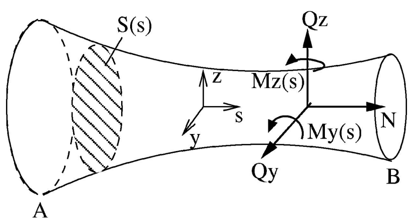

Figure 2 Beam with non-uniform thickness.

$$M_z(s) = \int_{S(s)} -y\sigma_{ss}(s, y, z)dydz \quad (9)$$

The balance equation imposes:

$$N(s) = N \quad (10)$$

$$M_y(s) = \bar{M}_y + sQ_z \quad (11)$$

$$M_z(s) = \bar{M}_z - sQ_y \quad (12)$$

where Q_y and Q_z are the shear forces, and $\bar{M}_y$ and $\bar{M}_z$ are the bending moment at the midpoint of the beam ($s = 0$).

The branch vector and the generalized stress associated to couple $c = \{A, B\}$, which is oriented from A to B, are

$$\underline{l}_c = \begin{pmatrix} l \\ 0 \\ 0 \end{pmatrix}, \quad \underline{f}_c = \begin{pmatrix} N \\ Q_y \\ Q_z \end{pmatrix}, \quad \underline{\hat{m}}_c = \begin{pmatrix} T \\ \bar{M}_y \\ \bar{M}_z \end{pmatrix}$$

where T is torsion.

The strength criterion writes:

$$|\sigma_{ss}(s, y, z)| \leq \sigma^* \quad (13)$$

where σ^* is the tensile yield stress. The ultimate yield strength domain of the beam, G^c, is the convex set of generalized stress $(\underline{f}_c, \underline{\hat{m}}_c) = (N, Q_y, Q_z, T, \bar{M}_y, \bar{M}_z)$ such that there exists $\sigma_{ss}(s, y, z)$ satisfying (7-11-12) for all s, and (13) for all (s, y, z). Of interest is the special case where (s, y) is a plane of material symmetry of the beam: $(y, z) \in S(s) \Leftrightarrow (y, -z) \in S(s)$, for all (s, y, z). Using uniaxial stress field $\sigma^{\sharp}_{ss}(s, y, z) = \sigma_{ss}(s, y, -z)$, it is easy to see that: $(\underline{f}_c, \underline{\hat{m}}_c) \in G^c \Leftrightarrow (\underline{f}_c, \underline{\hat{m}}_c)^{\sharp} \in G^c$, where $(\underline{f}_c, \underline{\hat{m}}_c)^{\sharp} = (N, Q_y, -Q_z, -T, -\bar{M}_y, \bar{M}_z)$ is the image of $(\underline{f}_c, \underline{\hat{m}}_c)$ by (s, y)-plane symmetry. Moreover, using $\sigma^{\flat}_{ss}(s, y, z) = \frac{1}{2}(\sigma^{\sharp}_{ss}(s, y, z) + \sigma_{ss}(s, y, z))$, it is clear that: $(\underline{f}_c, \underline{\hat{m}}_c) \in G^c \Rightarrow (\underline{f}_c, \underline{\hat{m}}_c)^{\flat} \in (G^c)^{\flat}$, where $(\underline{f}_c, \underline{\hat{m}}_c)^{\flat} = (N, Q_y, 0, 0, 0, \bar{M}_z) = \frac{1}{2}((\underline{f}_c, \underline{\hat{m}}_c) + (\underline{f}_c, \underline{\hat{m}}_c)^{\sharp})$ is the (s, y)-plane projection of $(\underline{f}_c, \underline{\hat{m}}_c)$, and $(G^c)^{\flat} \subset G^c$ is the subset of $(\underline{f}_c, \underline{\hat{m}}_c)$ in G^c such that the corresponding uniaxial stresses verify $\sigma^{\flat}_{ss} = \sigma_{ss}$.

The characterization of $(G^c)^{\flat}$ follows the method proposed by [8] for 2D beams. For any couple of real numbers (ϵ, χ), and for fixed s, the support function, $p(\epsilon, \chi; s)$, of the convex set of $(N(s), M_z(s))$ such that there exists σ_{ss} satisfying (7-8-9) and $\sigma^{\flat}_{ss} = \sigma_{ss}$ is introduced:

$$\epsilon N(s) + \chi M_z(s) = \int_{S(s)} (\epsilon - \chi y)\sigma_{ss}(s, y, z)dydz \leq \sigma^* \int_{S(s)} |\epsilon - \chi y|dydz \equiv p(\epsilon, \chi; s) \quad (14)$$

For $\chi = 0$, (14) is equivalent to $|N(s)| \leq |S(s)|\sigma^*$, where $|S(s)|$ is the area of section $S(s)$. For $\chi \neq 0$, one can divide by $|\chi|$ both members of inequality (14). Moreover, the limit case $\epsilon \to \pm\infty$ provides also $|N(s)| \leq A(s)\sigma^*$. This means that inequality (14) can be restricted without loss of generality to $\chi = \pm 1$.

Introducing the following notations:

$$-\frac{1}{2} \leq \tilde{s} = \frac{s}{l} \leq \frac{1}{2}, \quad \tilde{y} = \frac{y}{l}, \quad \tilde{z} = \frac{z}{l}$$

$$\tilde{N} = \frac{N}{l^2\sigma^*}, \quad \tilde{Q} = \frac{Q_y}{l^2\sigma^*}, \quad \tilde{M} = \frac{\bar{M}_z}{l^3\sigma^*} \quad (15)$$

and optimizing (14) for $\chi = \pm 1$ over all ϵ and all $\tilde{s}$, it is found that $(G^c)^{\flat}$ is completely characterized by:

$$-g(-\tilde{N}, -\tilde{Q}) \leq \tilde{M} \leq g(\tilde{N}, \tilde{Q}) \quad (16)$$

where

$$g(\tilde{N}, \tilde{Q}) = \operatorname*{Inf}_{\tilde{\epsilon} \in \mathbb{R}, |\tilde{s}| \leq \frac{1}{2}} \left\{ \int_{\tilde{S}(\tilde{s})} |\tilde{\epsilon} - \tilde{y}| d\tilde{y}d\tilde{z} - \tilde{\epsilon}.\tilde{N} + \tilde{s}.\tilde{Q} \right\} \quad (17)$$

If (s, z) is a plane of material symmetry of the beam: $(y, z) \in S(s) \Leftrightarrow (-y, z) \in S(s)$, for all (s, y, z), then $g(\tilde{N}, \tilde{Q}) = (g(-\tilde{N}, \tilde{Q})$. If (y, z) is a plane of material symmetry of the beam: $(y, z) \in S(s) \Leftrightarrow (y, z) \in S(-s)$, for all (s, y, z), then $g(\tilde{N}, \tilde{Q}) = g(\tilde{N}, -\tilde{Q})$. For uniform sections: $S(s) \equiv S$,

$$g(\tilde{N}, \tilde{Q}) = \operatorname*{Inf}_{\tilde{\epsilon} \in \mathbb{R}} \left\{ \int_{\tilde{S}} |\tilde{\epsilon} - \tilde{y}| d\tilde{y}d\tilde{z} - \tilde{\epsilon}.\tilde{N} \right\} - \frac{1}{2}|\tilde{Q}| \quad (18)$$

4.2. The non-symmetric material distribution

Consider a beam having circular Sections centered at $y = z = 0$, and the following non-symmetric material distribution:

$$|\tilde{S}(\tilde{s})| = \tilde{S}^m \left(1 + \Delta\tilde{S}\left(\frac{(\tilde{s} - \tilde{s}_0)^2}{\frac{1}{12} + \tilde{s}_0^2} - 1\right)\right) \quad (19)$$

where

- $|\tilde{S}(\tilde{s})|$ is the area of section $\tilde{S}(\tilde{s})$
- $\tilde{S}^m$ is the mean normalized Section area such that the volume of the beam is:

$$V = \tilde{S}^m.l^3$$

$$< |\tilde{S}| > = \int_{-1/2}^{1/2} |\tilde{S}(\tilde{s})| d\tilde{s} = \frac{1}{l^3} \int_{-l/2}^{l/2} |S(s)| ds = \tilde{S}^m$$

- $\Delta\tilde{S}$ is the relative variation of section

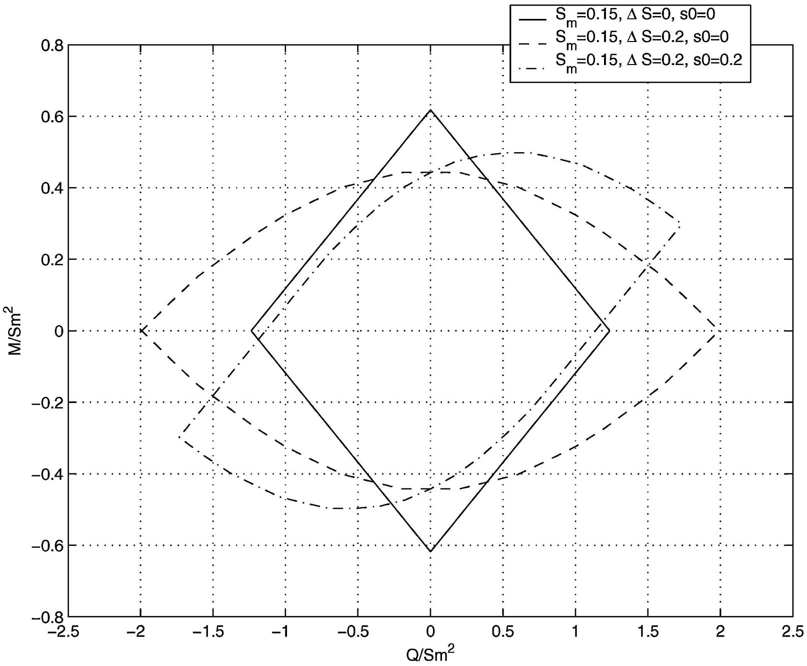

Figure 3 Strength domain of the beam in the $\tilde{N} = 0$ plane for three $(\Delta\tilde{S}, \tilde{s}_0)$.

– $-\frac{1}{2} \leq \tilde{s}_0 \leq \frac{1}{2}$ is the coordinate where beam section is minimum.

Parameter $\tilde{s}_0$ is a new defect parameter which describes the non-symmetry of the material distrubution in the cell edges. For $\Delta\tilde{S} = 0$, the material distribution is uniform and $\tilde{S}^m = \frac{S}{l^2}$. For $\Delta\tilde{S} \neq 0$, the minimum Section $\tilde{S}^m(1 - \Delta\tilde{S})$ is reached at $\tilde{s} - \tilde{s}_0$. In the special case $\tilde{s}_0 = 0$, the strut is symmetric.

Function $g(\tilde{N}, \tilde{Q})$, which is defined by (17), is numerically computed for three cases:

– The uniform case $\Delta\tilde{S} = 0$ for which $g(\tilde{N}, \tilde{Q})$ is given by (18).
– The symmetric case $\Delta\tilde{S} = 0.2$ and $\tilde{s}_0 = 0$.
– The non-symmetric case $\Delta\tilde{S} = 0.2$ and $\tilde{s}_0 = 0.2$.

Fig. 3 shows the strength domain of the beam, (16), for $N = 0$, in the plane of the normalized shear force $\frac{\tilde{Q}}{\tilde{S}^{m2}}$ and the normalized bending moment $\frac{\tilde{M}}{\tilde{S}^{m2}}$. It is observed that the effect of $\tilde{s}_0$ is to rotate and to deform the strength domain in the $N = 0$ plane. For $N = 0$, depending on the ratio of $\frac{\tilde{Q}}{\tilde{S}^{m2}}$ and $\frac{\tilde{M}}{\tilde{S}^{m2}}$, a non-symmetric material distribution will increase or decrease the plastic collapse strength of the beam. Fig. 4 shows the strength domain for $Q = 0$, in the plane of the normalized normal force $\frac{\tilde{N}}{\tilde{S}^m}$ and the normalized bending

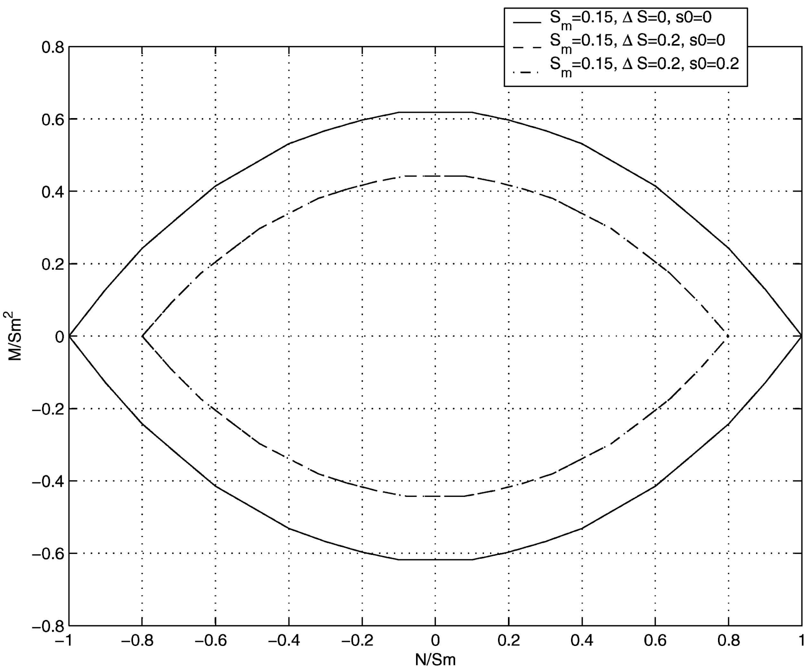

Figure 4 Strength domain of the beam in the $\tilde{Q} = 0$ plane for three $(\Delta\tilde{S}, \tilde{s}_0)$.

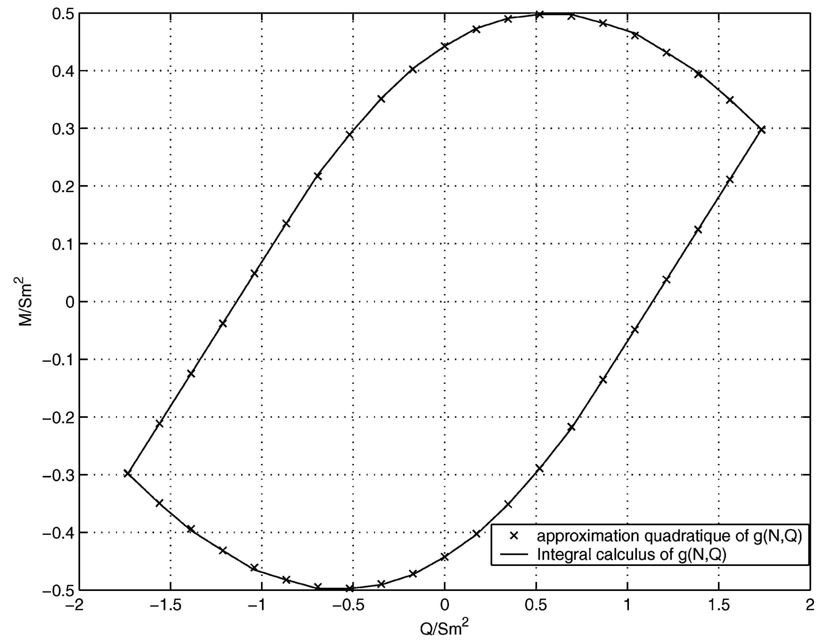

Figure 5 Quadratic approximation of the strength domain of the beam in the $\tilde{N} = 0$ plane for ($\tilde{S}_m = 0.15$, $\Delta\tilde{S} = 0.2$, $\tilde{s}_0 = 0.2$).

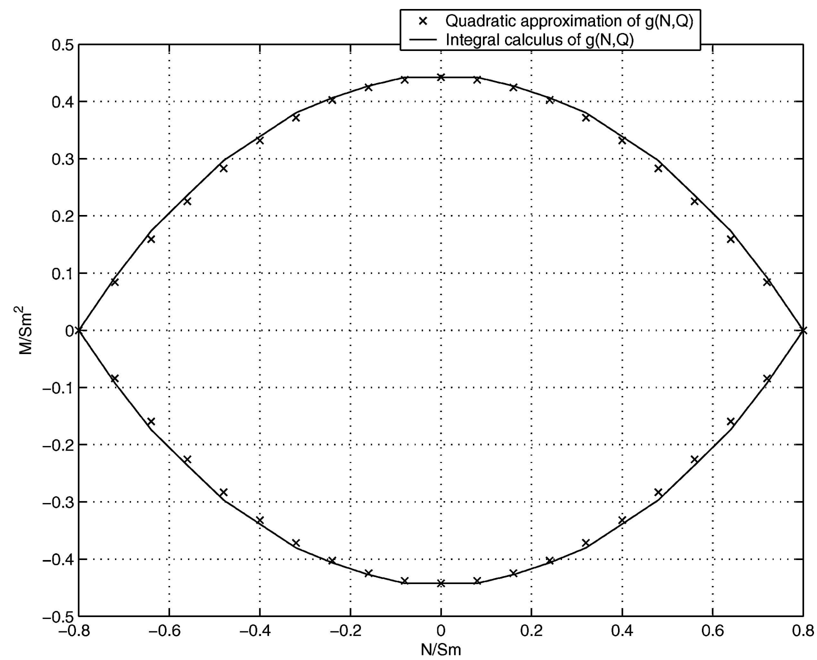

Figure 6 Quadratic approximation of the strength domain of the beam in the $\tilde{Q} = 0$ plane for ($\tilde{S}_m = 0.15$, $\Delta\tilde{S} = 0.2$, $\tilde{s}_0 = 0.2$).

moment $\frac{\tilde{M}}{\tilde{S}^{m2}}$. It is clear that $\tilde{s}_0$ has no influence on the plastic collapse strength for $Q = 0$.

Besides, these numerical results fit very well with the following quadratic expression for $g(\tilde{N}, \tilde{Q})$:

$$g_{app}(\tilde{N}, \tilde{Q}) = \operatorname*{Inf}_{|\tilde{s}| \leq \frac{1}{2}} \left\{ \alpha . \frac{\tilde{N}^2}{\tilde{R}(\tilde{s})} + \beta . \tilde{R}^3(\tilde{s}) + \tilde{s}\tilde{Q}) \right\} \tag{20}$$

where

– $\tilde{R}(\tilde{s})$ is the radius at the coordinate $\tilde{s}$
– $\beta = \frac{4}{3}$
– $\alpha = -\frac{4}{3\pi^2}$

Figs 5 and 6 show the comparaison between $g(\tilde{N}, \tilde{Q})$ and $g_{app}(\tilde{N}, \tilde{Q})$ for the non-symetric geometry ($\tilde{S}_m = 0.15$, $\Delta\tilde{S} = 0.2$, $\tilde{s}_0 = 0.2$) in the plane $N = 0$ (Fig. 5) and $Q = 0$ (Fig. 6).

5. Tetrakaidecahedral lattice

In this Section, the unit cell problem (6) is set and solved for a tetrakaidecahedral lattice of beams that are rigidely connected in vertices: the set $\mathcal{SA}(\underline{\underline{\Sigma}})$ of statically compatible generalized stresses is studied and the effective ultimate yield strength domain of the foam is analytically determined.

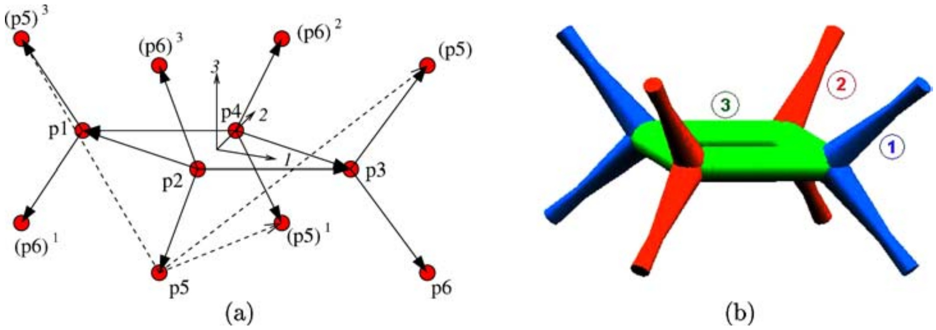

Figure 7 (a) Unit cell of a tetrakaidecahedral lattice, (b) Unit cell with $\Delta\tilde{S} \neq 0$, $\tilde{s}_0 \neq 0$.

5.1. Unit cell problem

The tetrakaidecahedral lattice corresponding to the well-known Kelvin partition of the 3D space is considered in this paper. Fig. 7a. It has been shown by Pradel [11] that this lattice is composed of 6 types of particles: $p_1, p_2, \ldots, p_6$ and of 12 oriented beams of length l that generate the lattice by periodicity according to the vectors which link p_5 to $(p_5)^1$, $(p_5)^2$ and $(p_5)^3$, or equivalently, p_6 to $(p_6)^1$, $(p_6)^2$ and $(p_6)^3$. The unit cell volume is $8\sqrt{2l^3}$ and the components in the reference of Fig. 7a of the branch vectors are:

$$\underline{l}_{\alpha_1,\alpha_2,\alpha_3} = \frac{l}{\sqrt{2}} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

where $\alpha_i \in \{-1, 0+1\}$, with $i = 1, 2, 3$, are such that $|\alpha_1| + |\alpha_2| + |\alpha_3| = 2$. It is assumed that the material distribution in the beams is such that (2, 3), (3, 1) and (1, 2) are three planes of material symmetry of the lattice. Therefore, there exist three types of beams as shown in Fig. 7b. Of interest are the macroscopic stress states $\underline{\underline{\Sigma}}$ which are invariant under the action of these plane symmetries. Therefore, the analysis is restricted to:

$$\underline{\underline{\Sigma}} = \begin{pmatrix} \Sigma_1 & 0 & 0 \\ 0 & \Sigma_2 & 0 \\ 0 & 0 & \Sigma_3 \end{pmatrix} \tag{21}$$

It is found that the periodic interaction forces and moments $(I_c)_{c\in\mathcal{C}}$ in $\mathcal{SA}(\underline{\underline{\Sigma}})$ which are invariant with respect to these symmetries are of the form:

$$\underline{f}_{\alpha_1,\alpha_2,\alpha_3} = 2l^2 \begin{pmatrix} \alpha_1\Sigma_1 \\ \alpha_2\Sigma_2 \\ \alpha_3\Sigma_3 \end{pmatrix}$$

$$\underline{\hat{m}}_{\alpha_1,\alpha_2,\alpha_3} = \begin{pmatrix} \alpha_2\alpha_3\bar{M}_1 \\ \alpha_1\alpha_3\bar{M}_2 \\ \alpha_1\alpha_2\bar{M}_3 \end{pmatrix}$$

where $\bar{M}_i$, with $i = 1, 2, 3$, are undetermined moments.

According to the analysis above, diagonal $\underline{\underline{\Sigma}}$ (21) in $\mathcal{G}^{\text{hom}}$ are such that there exist $\bar{M}_i$, with $i = 1, 2, 3$, verifying:

$$-g_i(-\tilde{N}_i, -\tilde{Q}_i) \leq \tilde{M}_i \leq g_i(\tilde{N}_i, \tilde{Q}_i) \tag{22}$$

where (N_i, Q_i) are the normal and shear force of the beam defined by $\alpha_1, \alpha_2, \alpha_3$ such that $\alpha_i = 0$ and the other two α being equal to +1. They are expressed in terms of $\underline{\underline{\Sigma}}$ as:

$$\begin{pmatrix} \tilde{N}_i \\ \tilde{Q}_i \end{pmatrix} = \frac{\sqrt{2}}{l\sigma^*}\underline{\underline{P}}_i \begin{pmatrix} \Sigma_1 \\ \Sigma_2 \\ \Sigma_3 \end{pmatrix} \tag{23}$$

where

$$\underline{\underline{P}}_1 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \quad \underline{\underline{P}}_2 = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\underline{\underline{P}}_3 = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}$$

Moreover, for $i = 1, 2, 3$, inequality (22) is compatible if:

$$g_i(\tilde{N}_i, \tilde{Q}_i) + g_i(-\tilde{N}_i, -\tilde{Q}_i) \geq 0 \tag{24}$$

5.2. Regular non-symmetric cells

The analytical model described above is utilized to investigate the effect of non-symmetric material distribution in the cell edges of regular tetrakaidecahedral lattice on the overall strength of these materials.

Recall that $\mathcal{G}^{\text{hom}}$ is the closed convex set of $\underline{\underline{\Sigma}}$ such that inequalities (24), with $i = 1, 2, 3$, are verified with $(\tilde{N}_i, \tilde{Q}_i)$ given by (23), and $g(\tilde{N}_i, \tilde{Q}_i)$ defined by (17) and approximated by (20). It is assumed that beams of types 1 and 2 have the same material distribution $((\tilde{S}^m)_i = \tilde{S}^m, (\Delta\tilde{S})_i = \Delta\tilde{S}, (\tilde{s}_0)_i = \tilde{s}_0)$ for $i = 1, 2$, and that beams of types 3 are uniform $((\tilde{S}^m)_3 = \tilde{S}_1(-\frac{1}{2}) = \tilde{S}_2(-\frac{1}{2}), (\Delta\tilde{S})_3 = 0, (\tilde{s}_0)_3 = 0)$. Fig. 7b. It should be emphasized that the relative density of this lattice si $\frac{3}{2\sqrt{2}}\tilde{S}^m(1 + \frac{\Delta\tilde{S}}{3}\frac{\frac{1}{6}+\tilde{s}_0}{\frac{1}{12}+\tilde{s}_0^2})$.

5.3. Results

Fig. 8 represents the yield surface in the $\Sigma_1 = \Sigma_2$ plane for $\tilde{S}^m = 0.15$ and three $(\Delta\tilde{S}, \tilde{s}_0) : (\Delta\tilde{S} = 0, \tilde{s}_0 = 0)$

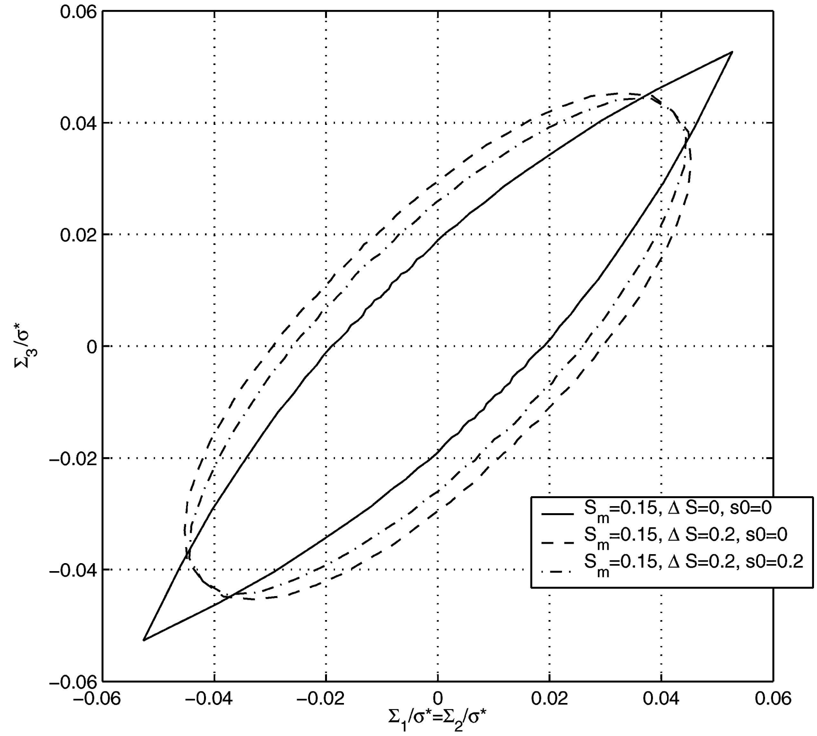

Figure 8 Yield surface in the $\Sigma_{11} = \Sigma_{22}$ plane for three $(\Delta\tilde{S}, \tilde{s}_0)$.

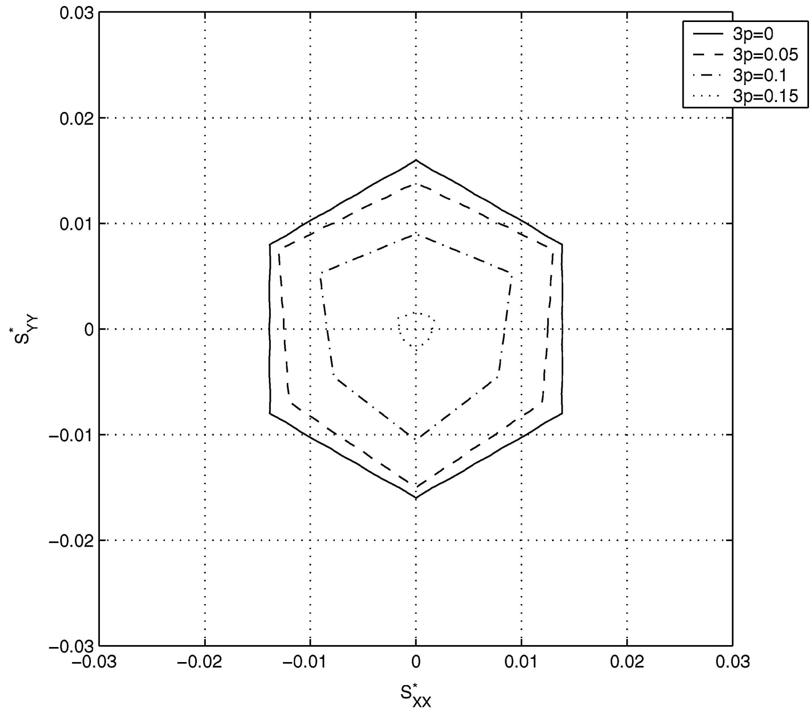

Figure 9 Yield surface in the $\Sigma_{11} + \Sigma_{22} + \Sigma_{33} = 3p$ plane for $\tilde{S}_m = 0.15$, $\Delta\tilde{S} = 0$, $\tilde{s}_0 = 0$ and four values of p.

(the uniform case), $(\Delta\tilde{S} = 0.2, \tilde{s}_0 = 0)$ and $(\Delta\tilde{S} = 0.2, \tilde{s}_0 = 0.2)$. It is observed that $\tilde{s}_0$ has a weak effect on the strength domain, whereas $\Delta\tilde{S}$ changes its shape. For $\Delta\tilde{S} \neq 0$, the strength domain area is not affected. On the other hand, the yield surface is more contracted in the $\Sigma_3 = \Sigma_1$ direction and it is more expanded in the perpendicular direction. Depending on the loading direction, the non-uniform material distribution is more or less resistant than the uniform one.

Figs. 9–11 show the yield surface in the $\Sigma_1 + \Sigma_2 + \Sigma_3 = 3p$ plane, for the same three selected values of $(\Delta\tilde{S}, \tilde{s}_0)$ and for four values of $p : 3p = 0, 3p = 0.05, 3p = 0.01$ and $3p = 0.015$. Parameter $\tilde{s}_0$ has a weak effect on the strength domain (Figs 10 and 11). It should be emphasized that parameter $\tilde{\Delta}S$ has the most significant effect on the overall strength (Figs, 9 and 10). For $p \leq 0.01$, the non-uniform material distribution is more resistant than the uniform one. For $p = 0$, 20% of relative variation of the section ($\tilde{\Delta}S = 0.2$) increases up to 56% the strength in the deviatoric stress direction $S_{XX} = 0$ and up to 64% in the stress direction $S_{YY} = 0$.

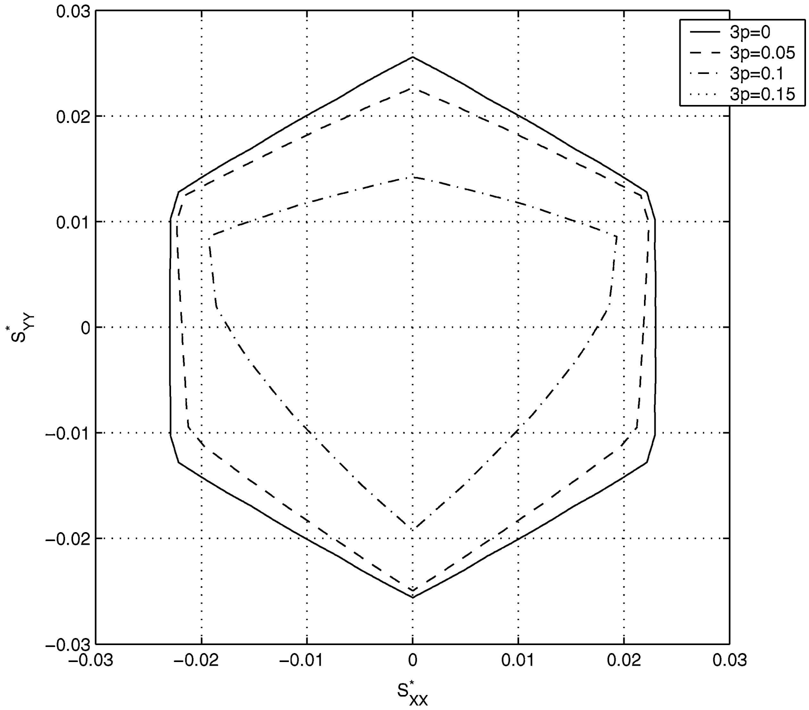

Figure 10 Yield surface in the $\Sigma_{11} + \Sigma_{22} + \Sigma_{33} = 3p$ plane for $\tilde{S}_m = 0.15$, $\Delta\tilde{S} = 0$, $\tilde{s}_0.2 = 0$ and four values of p.

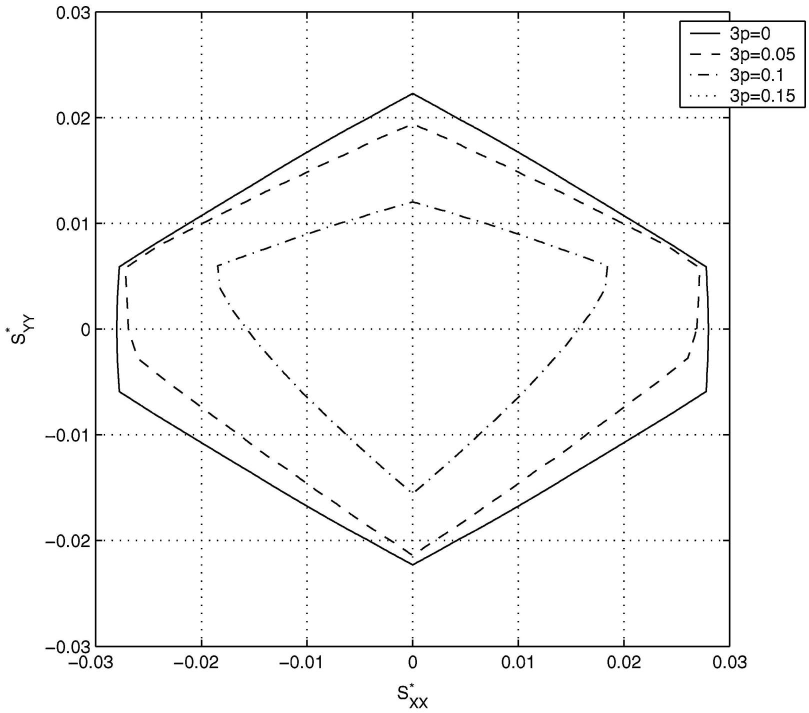

Figure 11 Yield surface in the $\Sigma_{11} + \Sigma_{22} + \Sigma_{33} = 3p$ plane for $\tilde{S}_m = 0.15$, $\Delta\tilde{S} = 0.2$, $\tilde{s}_0 = 0.2$ and four values of p.

6. Summary

A rigorous method for the homogenization of general elastoplastic periodic lattices has been applied to 3D cellular solids. Ultimate yield surface of a non-uniform 3D Euler-Bernouilli have been determined with this method. Results shows that defects have an influence on the shape of the plastic domain. Then, the general model have been used to solve unit cell problem for tetrakaidecahedral lattice. This systematic method is well adapted for non-symmetric material distribution. Defects of this type have been introduced and new results concerning nonsymmetric material distribution in the cell struts of the foam have been obtained. They show that the non-uniform material distribution increases the plastic collapse strength, except for particular loading directions. The non-symmetry of the material distribution has no significant influence on the yield surface.

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Received December 2004
and accepted April 2005